

DORIS

DAYS
2025

November, 3 and 5

Positioning

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November 05

Session 2 – Presentation 4

Positioning

Why is a precise (centimeter or even sub-centimeter) positioning of orbits and stations using the DORIS technique important?



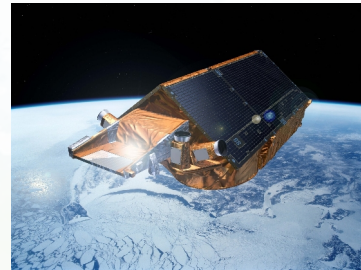
Positioning

Precise satellite orbits :

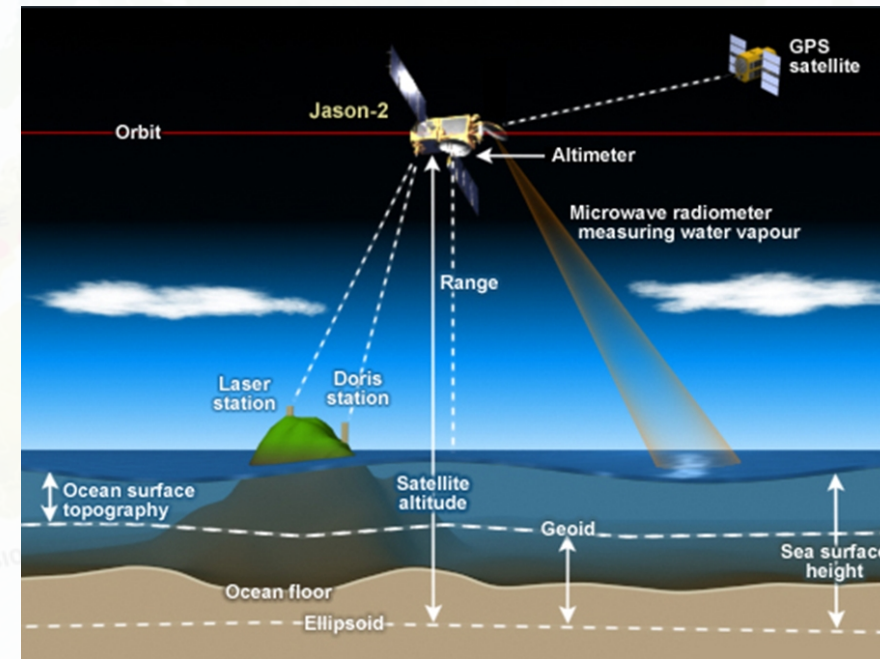
- Essential for satellite altimetry and ice sheet monitoring
 - All recent major altimetry missions carry a DORIS instrument to ensure precise orbit determination.
 - CryoSat-2, one of the key missions for studying ice sheet elevation, carries a DORIS instrument on board to ensure precise orbit determination.



Sentinel-6A
Image credit: NASA/JPL-Caltech



CryoSat-2
Image credit: ESA



Altimetry illustration

Image credit:

NOAA

- Can provide valuable information for refining atmospheric density models (e.g., DTM) through the estimation of correction parameters.

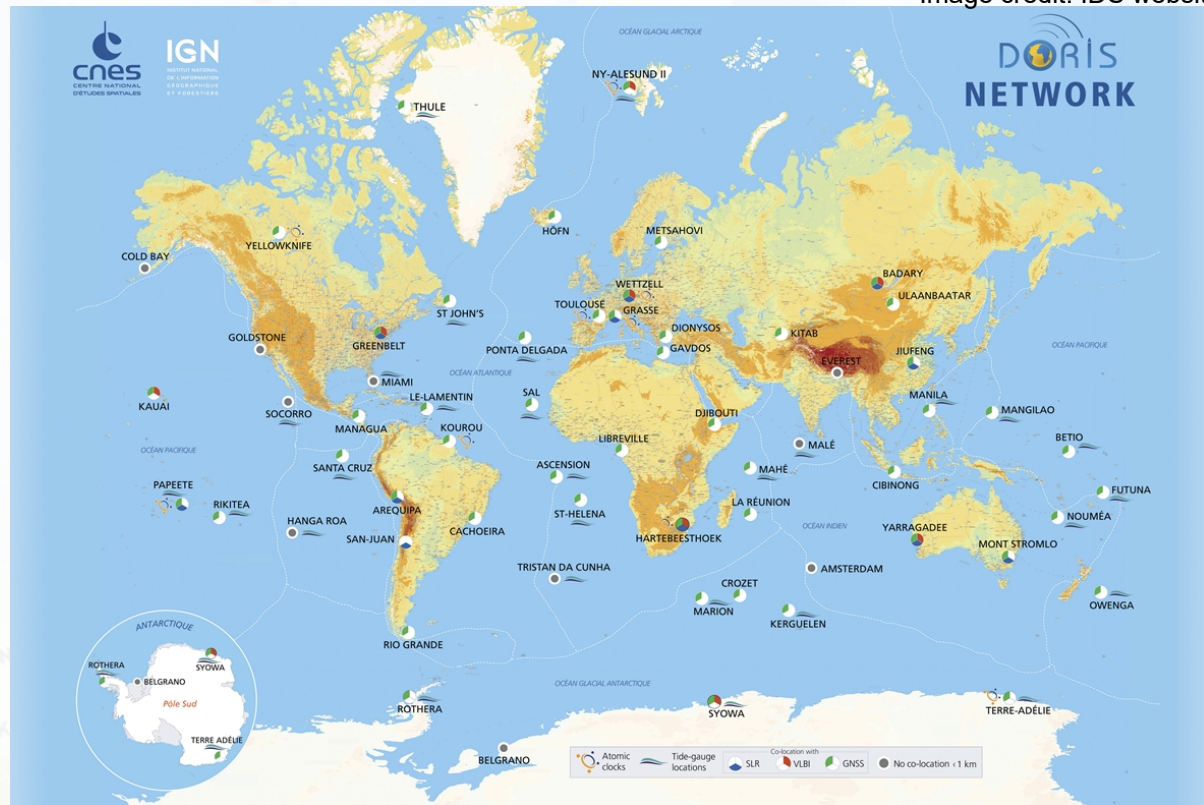
Positioning

Precise station positions :

- Local geophysical information (ground stability, tectonic plate velocities, etc.)
- Essential for precise orbit determination
- Essential for linking space geodesy techniques and mitigating network effects through a well-distributed station network and co-location sites, which is critical for realizing the International Terrestrial Reference Frame (ITRF)



Toulouse station coordinate time series
source : IDS Web Service



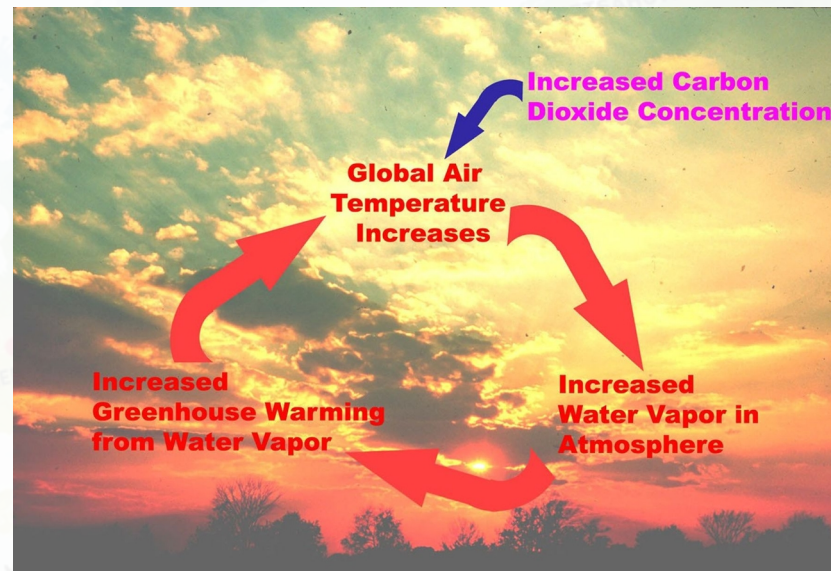
DORIS Network 2025 – Co-location with Other Space Geodesy Techniques

Image credit: IDS website

Positioning

Precise station positions & satellite orbits :

- Estimation of tropospheric parameters, directly linked to atmospheric water vapor content — an Essential Climate Variable (ECV) identified by the IPCC and GCOS, as water vapor is the most important greenhouse gas and a major driver of climate feedbacks.
- Estimation of ionospheric Total Electron Content (TEC) using DORIS dual-frequency measurements (similar to GNSS-based studies)



Positive Water Vapor Feedback Loop

Image credit: NASA Science

Overview

I. Mathematical Basis for Positioning

- I.1. Coordinates and reference system
- I.2. Cartesian frame and coordinates
- I.3. Frame transformations

II. DORIS Positioning Specifics

- II.1. Definition of the DORIS reference frame
- II.2. Sensitivity of measurements to frame parameters

III. Practical Positioning process : From observation equations to coordinates

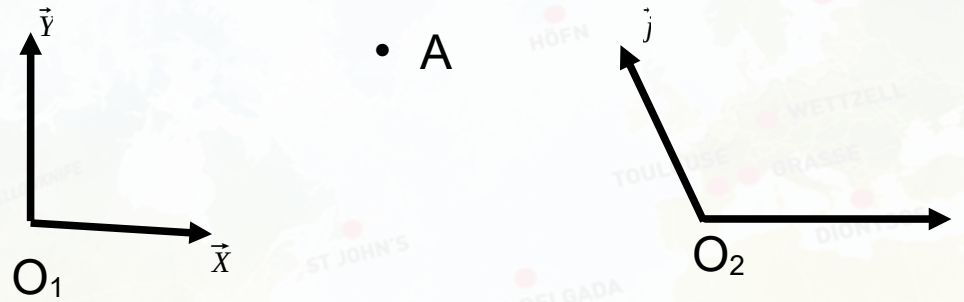
- III.1. Least-Squares Adjustment principles
- III.2. Linearization of observation equations
- III.3. Rank deficiency and Minimum Constraint Conditions (MCC)

Mathematical Basis for Positioning

- **Objective: Positioning in Space Over Time**

To determine the position of an object or station at different moments, we need a consistent mathematical framework to describe its location in space.

=> We need a reference system



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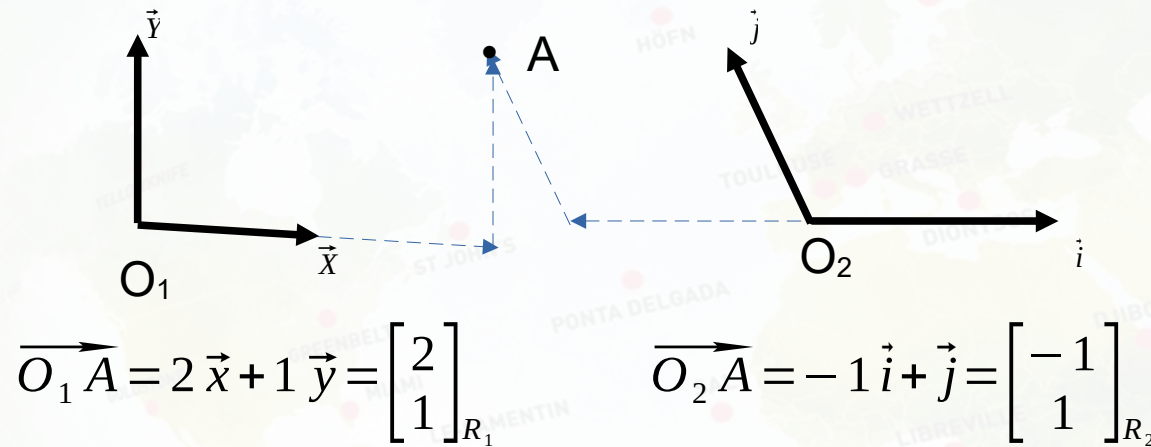


Mathematical Basis for Positioning

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Mathematical Basis for Positioning

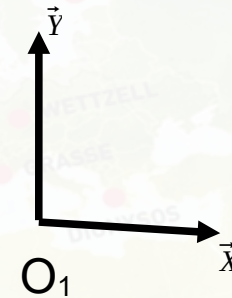
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=> We need a reference system

Positioning requires defining a reference frame, which provides the context in which coordinates are expressed. This frame includes:

- An origin point : O_1
- A set of N linearly independent vectors that spans an N-dimensional space forming a basis of the N-dimensional space : $(\vec{x}, \vec{y})$



The coordinates of point A represent the unique linear combination of the basis vectors that equals the vector from the origin to A.

Mathematical Basis for Positioning

- Cartesian frame and coordinates, in a 3D space :

A Cartesian reference frame consists of:

- An origin point, which serves as the zero point of the system
- Three mutually perpendicular axes (typically labeled x, y, and z in 3D)
- A set of orthonormal basis vector (**i**, **j**, and **k**), aligned with each axis and having unit length.

Cartesian coordinates :

The Cartesian coordinates of a point are a triplet of real numbers (x, y, z) that represent the unique linear combination of the basis vectors needed to reach the point from the origin : **OA** = $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

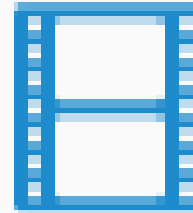
Mathematical Basis for Positioning

- **From one frame to another :**

$$X_{R_1} = S * R * X_{R_2} + T$$

with :

- X_{R_1} (resp. X_{R_2}) coordinates of X in frame R_1 (resp. R_2)
- T the translation vector from the origin of R_2 to the origin of R_1
- s the scale factor (can be scalar or a diagonal matrix for anisotropic scaling)
- R the rotation matrix that aligns the axes of R_2 to those of R_1



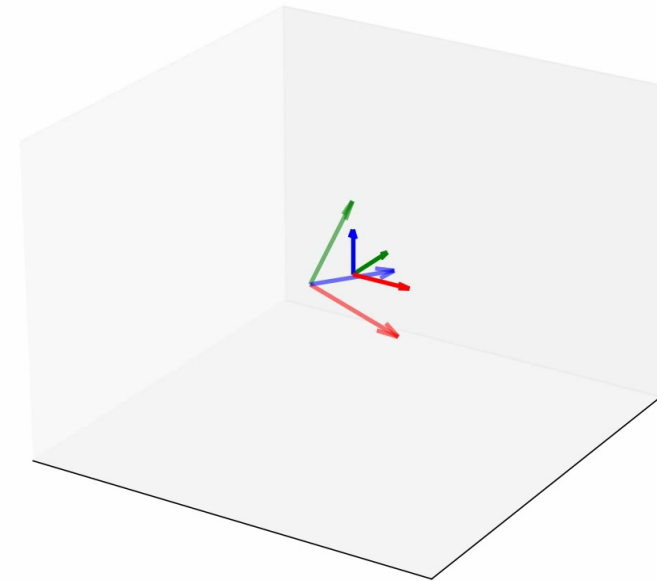
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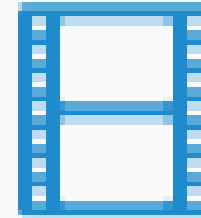
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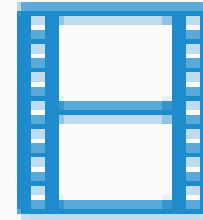
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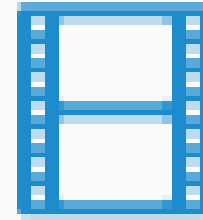
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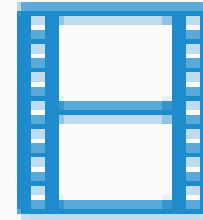
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Mathematical Basis for Positioning

Special Case: When Reference Frames Are Close, in 3D

In the case where frames are close, meaning :

- Their axes are nearly aligned (Rotation angles $< 10^{-5}$ rad)
- Their origins are close to each other (Translations < 100 m)
- Their scales are almost identical (Scale factor $D = s^{-1} < 10^{-5}$)

We can linearize the transformation equation using a first-order approximation of R :

$$X_{R_1} = X_{R_2} + T + DX_{R_2} + LRX_{R_2} \text{ with } LR = \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix}$$

$$\text{and for } X_{R_i} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{R_i} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}_{ref} + \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}_{R_i} \text{ with } i=\{1,2\} : \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}_{R_1} = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}_{R_2} + \begin{bmatrix} 1 & 0 & 0 & x_0 & 0 & z_0 & -y_0 \\ 0 & 1 & 0 & y_0 & -z_0 & 0 & x_0 \\ 0 & 0 & 1 & z_0 & y_0 & -x_0 & 0 \end{bmatrix} \begin{bmatrix} T_x \\ T_y \\ T_z \\ D \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$\delta X_{R_1} = \delta X_{R_2} + A\theta$ with θ the 7 transformation parameters.

This corresponds to the **7-parameter Helmert transformation**.

DORIS Positioning Specifics

- **The DORIS reference system in geodesy**

Correspond to **ITRS** : International Terrestrial Reference System

Origin : Earth center of mass, including oceans and atmosphere.

Scale : Consistent with the SI definition of the meter, based on the speed of light in vacuum

Orientation :

- The Z-axis points towards the International Reference Pole (mean rotation axis of the Earth)
- The X-axis points towards the intersection of the equator and the reference meridian (near Greenwich)
- The Y-axis completes the right-handed coordinate system.

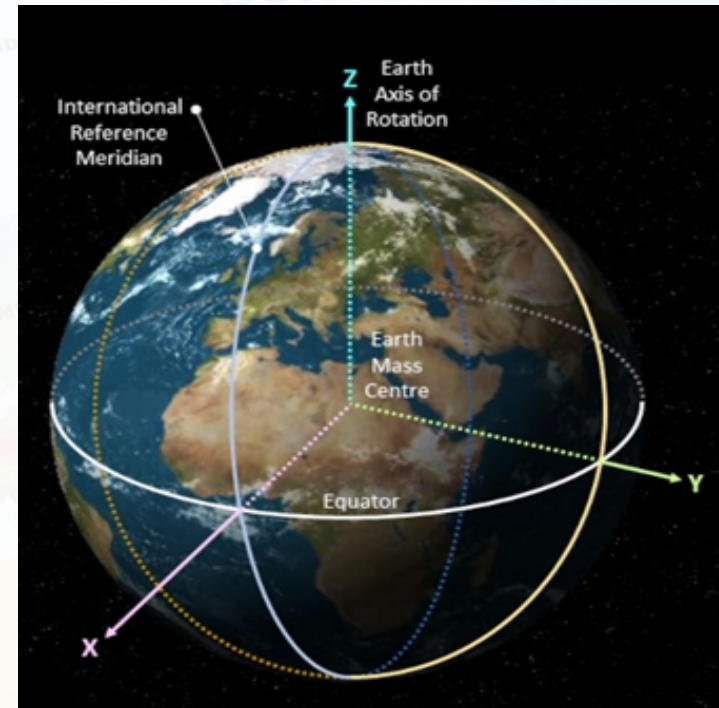


Image credit : Geoscience Australia
<https://geoscienceaustralia.github.io/ginan/images/ITRF-75pc.png>

DORIS Positioning Specifics

- Sensitivity of DORIS measurements/models to frame parameters**

Origin : The origin of the reference frame corresponds to one of the foci of the DORIS satellite orbits. Therefore, in theory, the measurements from the DORIS technique are **sensitive** to the definition of this origin.

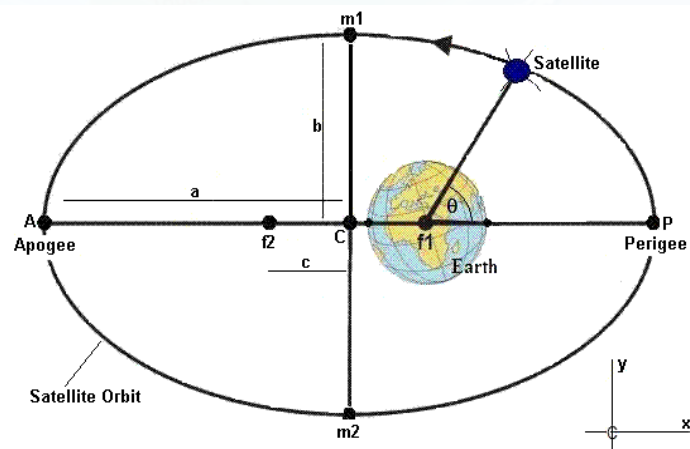


Image credit : Australian Space Academy

Scale : Sensitive to the SI definition of the meter. DORIS uses radio signals that travel at the speed of light in vacuum.

Orientation : Must be conventionally defined !

How is the orientation defined ? We align the DORIS frame to a reference frame such as ITRF, ensuring no net global rotation between the two frame.

Practical Positioning process

- **From observation equations to coordinates**

The DORIS observation equations link the observations to parameters like the position coordinates X of stations, among others.

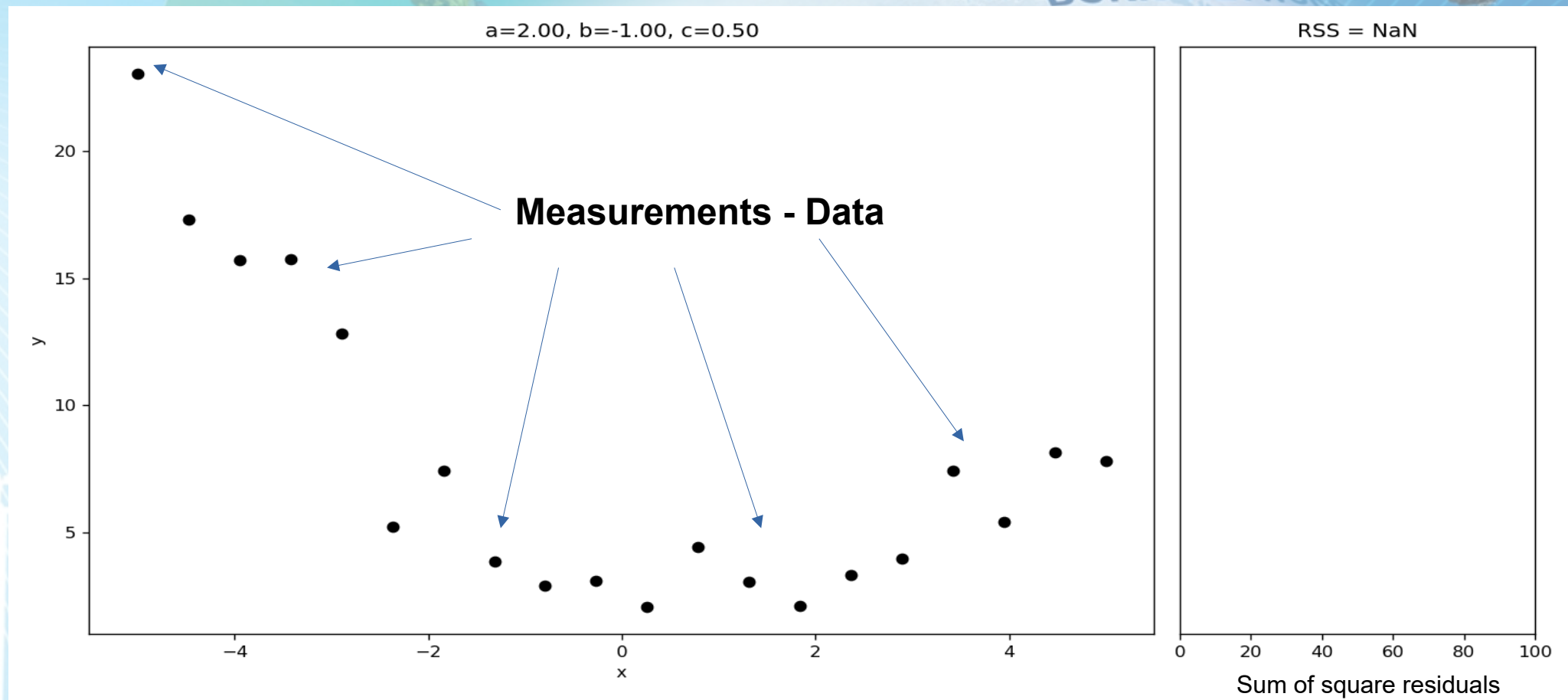
$$Y = f(X, \text{others})$$

The goal is to estimate X with an overdetermined system — more observations than unknowns, with a non-linear model f .

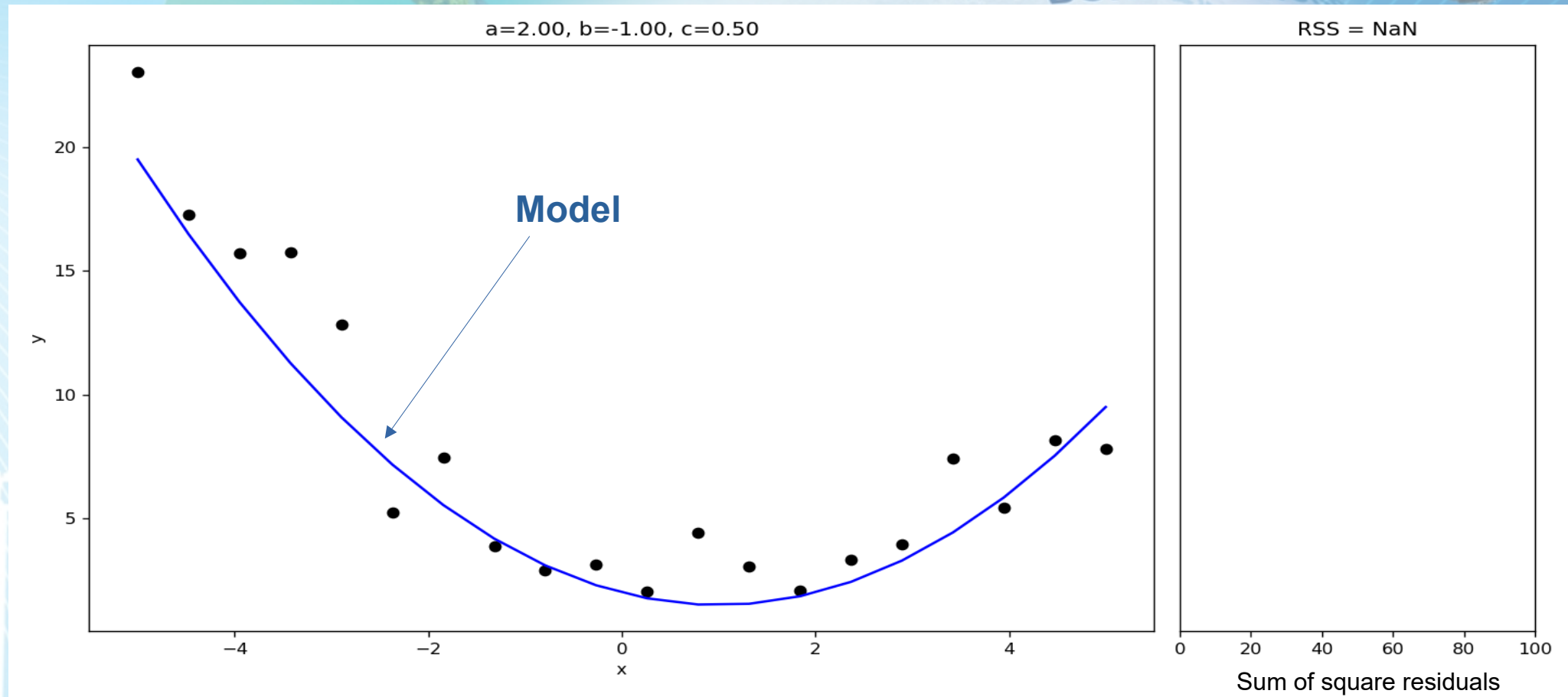
Many mathematical methods exist to achieve this. In geodesy, the most commonly used are the least squares method and the Kalman filter.

Here, we will focus on the classical **least squares method**.

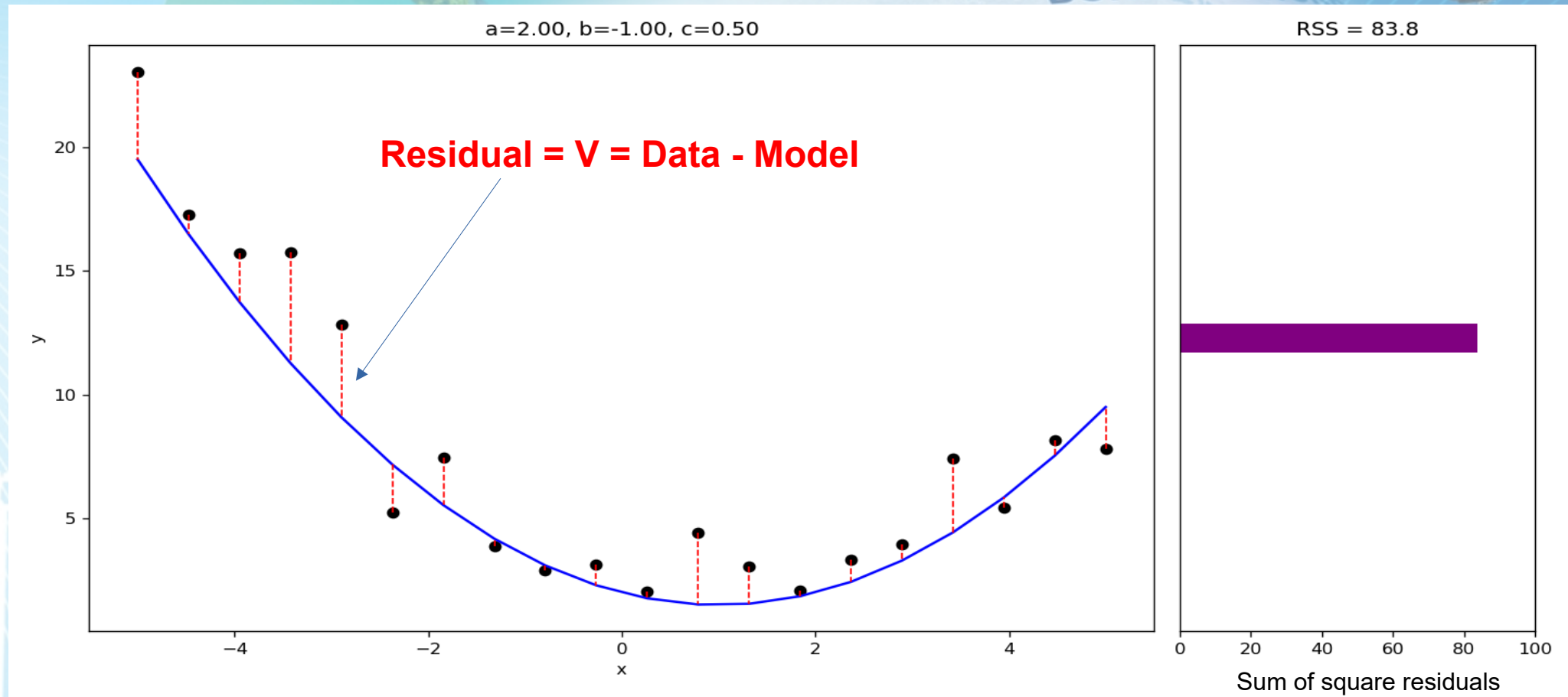
Least square adjustment



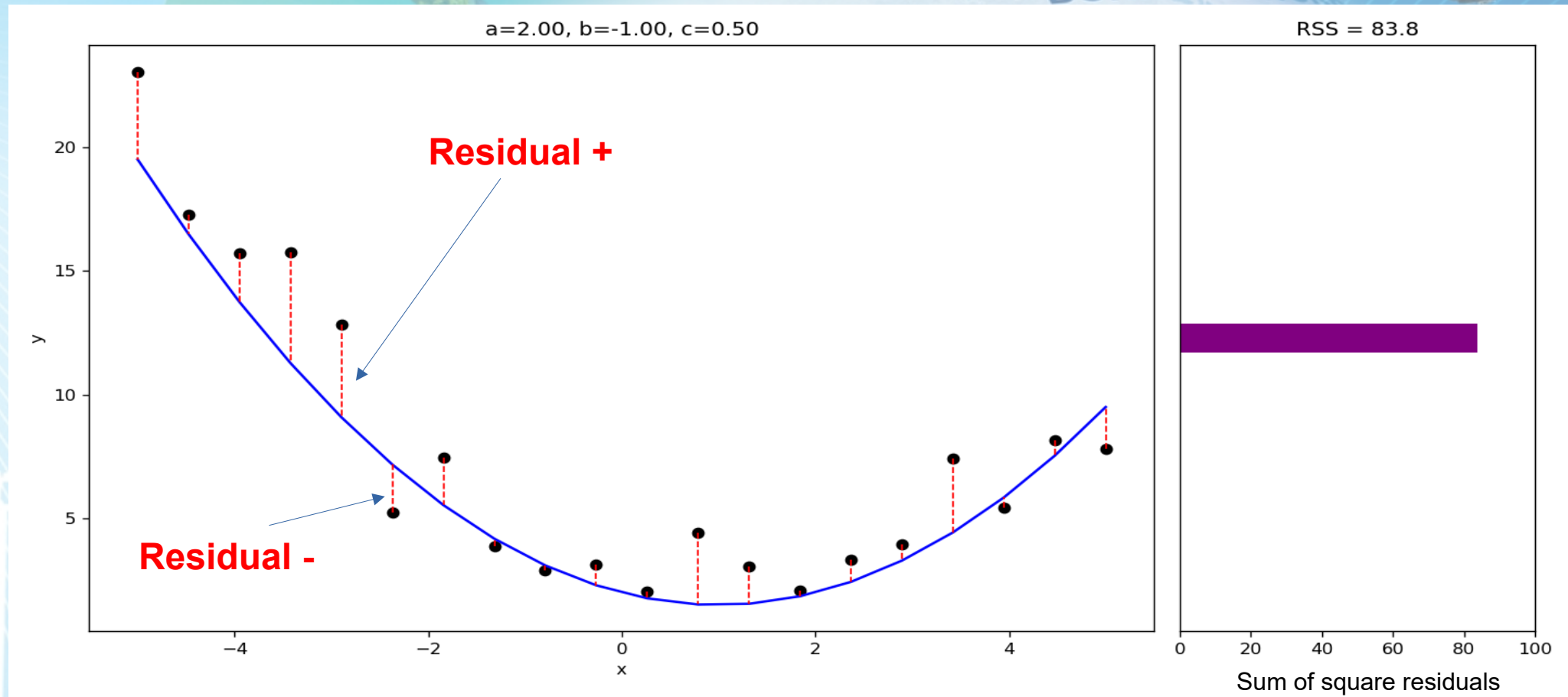
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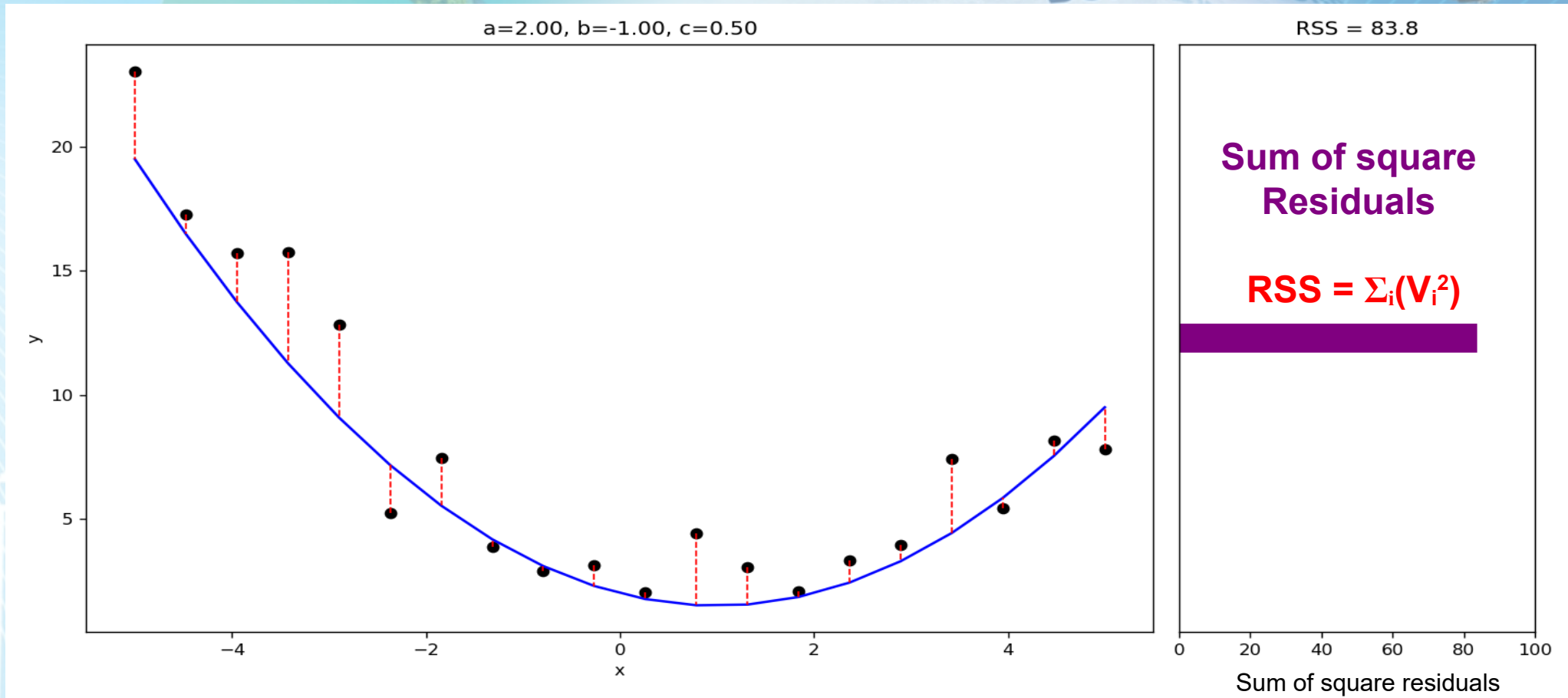
Least square adjustment



Least square adjustment



Least square adjustment



Least square adjustment

Minimize RSS

Sum of square residuals

Least square adjustment

Linear method

Assume the problem can be written in matrix form as :

$$\mathbf{Y} = \mathbf{AX} (\Sigma_Y)$$

Where :

$\mathbf{Y}$ the vector of n observations.

$\mathbf{A}$ the design matrix.

$\mathbf{X}$ the vector of p parameters to be estimated.

Σ_Y the associated variance-covariance matrix of the observations (often diagonal), known up to a scale factor σ_0^2 .

Least square adjustment

Linear method

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Solution :

$$\hat{\mathbf{X}} = (\mathbf{A}^T \Sigma_Y^{-1} \mathbf{A})^{-1} \mathbf{A}^T \Sigma_Y^{-1} \mathbf{Y}$$

$$\Sigma_X = \sigma_0^2 N^{-1} \text{ where the normal matrix } N = \mathbf{A}^T \Sigma_Y^{-1} \mathbf{A}, \sigma_0^2 = \frac{\mathbf{V}^T \Sigma_Y^{-1} \mathbf{V}}{n - p}$$

and the residuals $\mathbf{V} = \mathbf{Y} - \mathbf{A} \hat{\mathbf{X}}$.

Practical Positioning process

- **From observation equations to coordinates**

The DORIS observation equations link the observations to parameters like the position coordinates X of stations, among others.

$$Y = f(X, \text{others})$$

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HOW TO USE LEAST SQUARE METHOD WITH NON LINEAR MODEL ?

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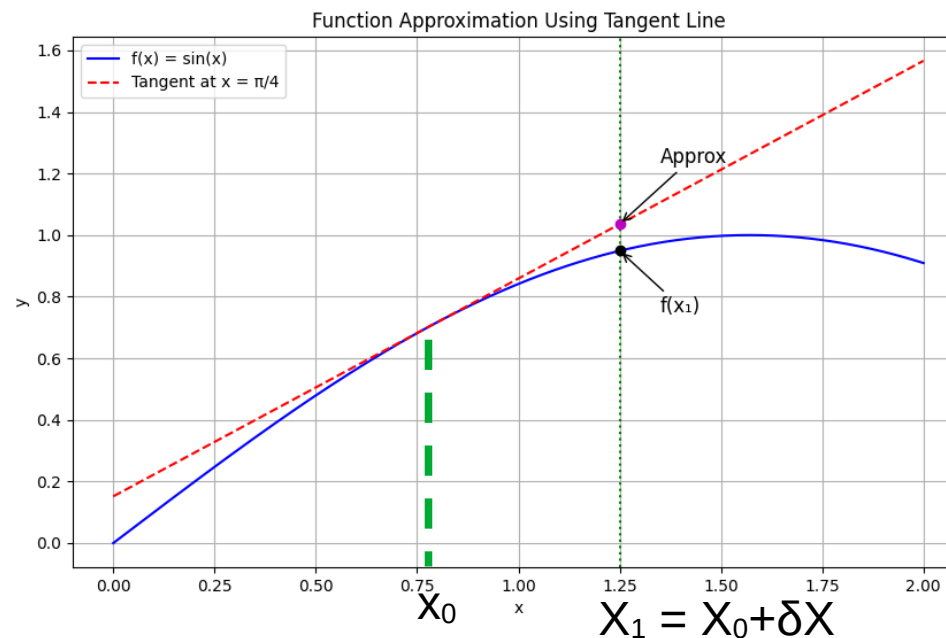
=> The model is linearized around a priori estimates of the parameters.

Practical Positioning process

Example : A simple case of a non linear function of two parameters.

Linearization around (x_0, y_0) :

$$f(x, y) \approx f(x_0, y_0) + (x - x_0) \frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0) \frac{\partial f}{\partial y}(x_0, y_0) = f(x_0, y_0) + \delta x \frac{\partial f}{\partial x}(x_0, y_0) + \delta y \frac{\partial f}{\partial y}(x_0, y_0)$$

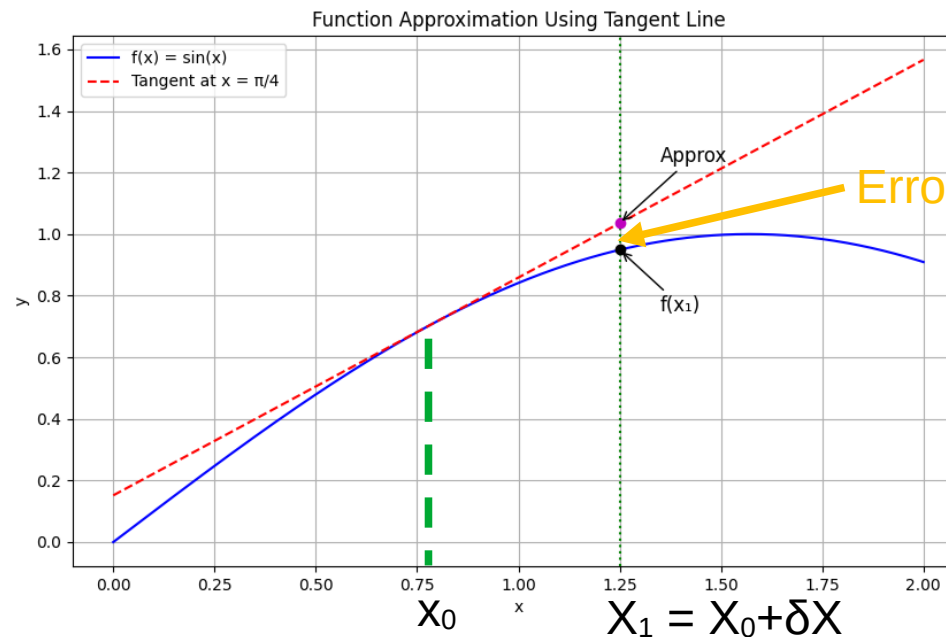


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Practical Positioning process

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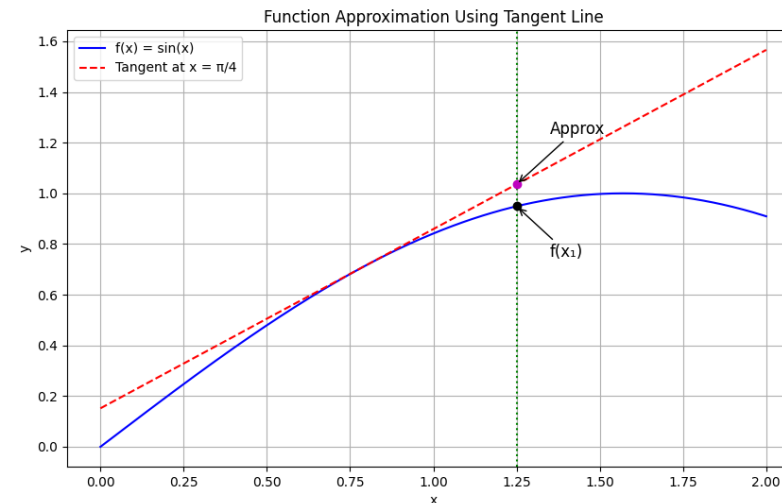
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For one observation y_i :

$$y_i = f_i(x, y) \Rightarrow y_i - f_i(x_0, y_0) = \begin{bmatrix} \frac{\partial f_i}{\partial x}(x_0, y_0) & \frac{\partial f_i}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$

$$Y = \begin{bmatrix} y_1 - f_1(x_0, y_0) \\ \vdots \\ y_n - f_n(x_0, y_0) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x_0, y_0) & \frac{\partial f_1}{\partial y}(x_0, y_0) \\ \vdots & \vdots \\ \frac{\partial f_n}{\partial x}(x_0, y_0) & \frac{\partial f_n}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$



Practical Positioning process

Example : A simple case of a non linear function of two parameters.

Linearization around (x_0, y_0) :

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Pseudo-observations

Correction vector
Adjustment vector
Update vector
Parameter corrections
...

Design matrix, Model matrix, Jacobian matrix

Practical Positioning process

- From observation equations to coordinates

The DORIS observation equations link the observations to parameters like the position coordinates X of stations, among others.

$$Y = f(X, \text{others})$$

The model need to be linearized around a priori estimates of the parameters.

Solution : iterative process

$$\hat{X} = (A^T \Sigma_Y^{-1} A)^{-1} A^T \Sigma_Y^{-1} Y$$

$$\Sigma_X = \sigma_0^2 N^{-1} \quad \text{where the normal matrix } N = A^T \Sigma_Y^{-1} A, \quad \sigma_0^2 = \frac{V^T \Sigma_Y^{-1} V}{n - p}$$

and the residuals $V = Y - A \hat{X}$

This method requires N to be invertible.

Practical Positioning process

- **From observation equations to coordinates**

N is non invertible when we have :

- **Fully correlated parameters.** One (or more) parameter is a linear combination of the others : The system is UNSOLVABLE ! Remove the correlated parameter(s) !
- **Lack of information** provided by the observations to estimate all parameters, involving a **rank deficiency**. The system needs to be constrained. In geodesy, this situation is very common.

The DORIS equation system needs to be constrained as soon as we estimate orbits, Earth Orientation Parameters (EOPs), and station coordinates. **We must define the frame orientation!** The normal matrix then exhibits a rank deficiency of 3, corresponding to the three rotational degrees of freedom.

However, if we only estimate station coordinates (with EOPs and orbits fixed), there is no rank deficiency, since the fixed EOPs and orbit implicitly define the orientation of the reference frame in accordance with the models.

Practical Positioning process

DORIS system usual constraints when estimating all parameters:

- Loose constraints :

$X = 0$ (σ) with σ being much larger than the possible range of the parameter X .

Example : Station coordinate correction wrt. ITRF2020. For DORIS, this correction is approximately 1cm.

A σ of 1m or more represents a loose constraint.

As these constraints do not allow to define the orientation but allow the inversion of N , we speak about a loose constrained solution.

- Minimal constraints :

In the case of DORIS, three constraints are applied to address the three rank deficiencies. These are referred to as **NNR constraints (No Net Rotation)**, as they mathematically express that the DORIS frame must share the same orientation as a reference frame, through a 7-helmert parameter transformation.

Since we apply three constraints to address the three rank deficiencies, these constraints are considered minimal, as they represent the minimum number required to invert matrix N .

Practical Positioning process

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In the case of DORIS, three constraints are applied to address the three rank deficiencies. These are referred to as NNR constraints (No Net Rotation), as they mathematically express that the DORIS frame must share the same orientation as a reference frame, through a 7-helmert parameter transformation.

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}_{R_1} = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}_{R_2} + \begin{bmatrix} 1 & 0 & 0 & x_0 & 0 & z_0 & -y_0 \\ 0 & 1 & 0 & y_0 & -z_0 & 0 & x_0 \\ 0 & 0 & 1 & z_0 & y_0 & -x_0 & 0 \end{bmatrix} \begin{bmatrix} T_x \\ T_y \\ T_z \\ D \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \quad \begin{matrix} \text{7-Helmert} \\ \text{transformation} \end{matrix}$$

Specific case :

DORIS frame = R1

Reference frame in which the a priori coordinates are expressed = R2

$$X_{Ref} = X_{R_2} = X_0 \Rightarrow \delta X_{R_2} = 0$$

$$X_{DORIS} = X_{R_1} = X_0 + \delta X_{R_1}$$

Practical Positioning process

DORIS system usual constraints when estimating all parameters:

- Minimal constraints :

$$\delta X_{DORIS} = \begin{bmatrix} \delta x_{sta1} \\ \delta y_{sta1} \\ \delta z_{sta1} \\ \vdots \\ \delta x_{staN} \\ \delta y_{staN} \\ \delta z_{staN} \end{bmatrix}_{DORIS} = \begin{bmatrix} 1 & 0 & 0 & x_0^{sta1} & 0 & z_0^{sta1} & -y_0^{sta1} \\ 0 & 1 & 0 & y_0^{sta1} & -z_0^{sta1} & 0 & x_0^{sta1} \\ 0 & 0 & 1 & z_0^{sta1} & y_0^{sta1} & -x_0^{sta1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & x_0^{staN} & 0 & z_0^{staN} & -y_0^{staN} \\ 0 & 1 & 0 & y_0^{staN} & -z_0^{staN} & 0 & x_0^{staN} \\ 0 & 0 & 1 & z_0^{staN} & y_0^{staN} & -x_0^{staN} & 0 \end{bmatrix} \begin{bmatrix} T_X \\ T_Y \\ T_Z \\ D \\ R_1 \\ R_2 \\ R_3 \end{bmatrix} = B \Theta$$

$$\longrightarrow (B^T B)^{-1} B^T \delta X_{DORIS} = C \delta X_{DORIS} = \Theta$$

NNR constraint equations: $R_i = 0, \{1,2,3\}$ (no global rotations between DORIS frame and Ref. frame)
Let **D** be the matrix containing the last three rows of **B**

$$\longrightarrow D \delta X_{DORIS} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Key points

- Accurate positioning matters (cm $\Rightarrow$ mm) for geophysic application
- Mathematical Reminder: Coordinates are linked to a reference frame
- DORIS measurements are sensible to geocenter motion and scale of the frame
- Estimation Method : Least Squares & NNR Constraints in DORIS